Generalized Null Lagrangians for Equations with Special Function Solutions

Atharva A. Dange,[1, a)] Lesley C. Vestal,[1, b)] and Zdzislaw E. Musielak[1, c)]

[1] *Department of Physics, University of Texas at Arlington, Arlington, TX 76019, USA*

[a)]Corresponding author: atharvaatul.dange@mavs.uta.edu
[b)]lesley.vestal@uta.edu
[c)]zmusielak@uta.edu

Abstract. A method to derive general standard and null Lagrangians for second-order differential equations whose solutions are special function of mathematical physics is presented. The general null Lagrangians are used to find the corresponding general gauge functions. All derived Lagrangians are new and in special cases they reduce to those published in the literature. The obtained results are applied to the Bessel, Hermite and Legendre equations, which have many applications in physics, applied mathematics and engineering.

INTRODUCTION

Second-order ordinary differential equations (ODEs), whose solutions are given in terms of special functions (SFs) of mathematical physics [1,2], have many important applications in physics and applied mathematics as shown in standard textbooks (e.g., [3]). For several of these ODEs, Lagrangians were previously constructed [4-6]. More recently, the standard and non-standard Lagrangians for the ODEs with the SF solutions were derived [7]. In this paper, a method is developed to derive the general standard Lagrangians (SLs) as well as the so-called general null Lagrangians (NLs), which identically satisfy the Euler-Lagrange (E-L) equation and can also be expressed as the total derivative of a scalar function, also called a gauge function [8]. The SLs depend on the square of the first derivative of the dependent variable (kinetic energy-like term) and the square of the dependent variable (potential energy-like term). The NLs were studied in mathematics [9,10] and have also applied to elasticity [11], and Newtonian mechanics where they were used to introduce forces [12]. However, the role of the NLs in ODEs with the SF solutions has not yet been fully explored; doing so is the main aim of this paper. Our choice to focus on these ODEs is justified by their many physical applications familiar to graduate and undergraduate science students. Our results are applied to the Bessel, Hermite and Legendre equations as these specific SFs are used in many physical applications. Therefore, the presented results should be of interest to physicists, applied mathematicians and engineers.

GENERAL STANDARD AND NULL LAGRANGIANS

Generalization

Let $\hat{D} = d^2/dx^2 + B(x)\, d/dx + C(x)$ be a linear operator whose coefficients *B(x)* and *C (x)* are ordinary and smooth (C^∞) functions. If $\hat{D}$ acts on *y(x),* which is also ordinary and smooth, then the resulting ODE can be written in the following explicit form

$$y'' + B(x)y' + C(x)y = 0. \tag{1}$$

By specifying the coefficients *B(x)* and *C (x),* all ODEs with the SF solutions are obtained and for these equations we derive the SLs and NLs. Let L_n be a null Lagrangian and L_s be a standard Lagrangian that is used to derive an ODE with the SF solutions. The NLs described above imply that the total Lagrangian L_{tot} given by $L_{tot} = L_s + L_n$ leads to the same ODE as using L_s only. In other words, the addition of L_n does not change the outcome when the E-L is applied. In this paper, we develop a general method to drive the standard Lagrangians together with a new family of NLs that are used to find the corresponding gauge functions.

The starting point of this method is to consider the general Lagrangian:

$$L(y', y(x), x) = \tfrac{1}{2} f_1(x) y'^2 + \tfrac{1}{2} f_2(x) y y' + \tfrac{1}{2} f_3(x) y^2 \tag{2}$$

where $f_1(x)$, $f_2(x)$, and $f_3(x)$ are ordinary and smooth functions to be determined. This Lagrangian depends on the square of the first derivative of the dependent variable (kinetic energy-like term), the square of the dependent variable (potential energy-like term) and on the mixed term with the dependent variable and its derivative. Substituting the above Lagrangian into the E-L equation, we obtain:

$$y'' + \left(\tfrac{f_1'}{f_1}\right) y' + \tfrac{1}{f_1}\left(\tfrac{1}{2} f_2' - f_3\right) y = 0. \tag{3}$$

Comparing (1) and (3) we get $B(x) = \frac{f_1'}{f_1}$ and $C(x) = \frac{1}{f_1}\left(\frac{1}{2} f_2' - f_3\right)$, which gives $f_1 = c_1 e^{\int B(x)dx} = c_1 E_s$ and $f_3 = \frac{1}{2} f_2' - C(x) \cdot f_1 = \frac{1}{2} f_2' - C(x) \cdot (c_1 E_s)$, where c_1 is the integration constant and $E_s = e^{\int B(x)dx}$. Substituting $f_1(x)$ and $f_3(x)$ into $L(y', y, x)$, we find:

$$L(y', y, x) = L_s(y', y, x) + L_n(y', y, x) \tag{4}$$

where

$$L_s(y', y, x) = \tfrac{1}{2} c_1 E_s(x) \left[y'^2(x) - C(x) y^2(x)\right] \tag{5}$$

and

$$L_n(y', y'x) = \tfrac{1}{2} y(x) \left[f_2(x) y'(x) + \tfrac{1}{2} f_2'(x) y(x)\right] \tag{6}$$

with $L(y', y, x)$ being a combination of the general standard Lagrangian $L_s(y', y, x)$ and the general null Lagrangian $L_n(y', y, x)$. It must be noted that $L_s(y', y, x)$ generalizes the Caldirola-Kanai (CK) Lagrangian [13,14] and it reduces to the CK Lagrangian when $B(x) = b = const.$ and $C(x) = c = const$; this SL also describes a harmonic oscillator with time dependent mass and spring constant.

General Gauge Functions

Having obtained the general null Lagrangian, we now derive its general gauge function φ using:

$$L_{null} = \tfrac{1}{4} f_2'(x) y^2(x) + \tfrac{1}{2} f_2(x) y(x) y'(x) = \frac{d\varphi}{dx} = \frac{\partial\phi}{\partial x}\Big|_{y=c} + \frac{\partial\phi}{\partial y}\Big|_{x=c} \cdot y'(x). \tag{7}$$

Which gives,

$$\varphi = \tfrac{1}{4} f_2(x) y^2(x). \tag{8}$$

With $f_2(x)$ being arbitrary, the following three cases may be considered:

i) $f_2 = 0 \quad \rightarrow L_S = L_{S,min}$ (*trivial case*)

ii) $f_2 = constant \rightarrow L_{mid} = L_{S,mid} + L_{n,mid}$

iii) $f_2 = f_1' \quad \rightarrow L_{max} = L_{S,max} + L_{n,max}$

Substituting each of the cases in our gauge equation (8), we get our three-gauge functions respectively:

i) $\varphi = \frac{1}{4} f_2(x) y^2(x) = 0$ *No gauge function*
ii) $\phi = \frac{1}{4} f_2(x) y^2(x) = \frac{1}{4} \cdot c \cdot y^2(x) = c_2 y^2(x)$ *Variable gauge function*
iii) $\phi = \varphi_{max} = \frac{1}{4} c_1 \cdot E_s(x) \cdot B(x) \cdot y^2(x)$ *Max variable gauge function*

APPLICATIONS

Applications of our results to selected ODEs with the SF solutions are summarized in the following table.

TABLE 1.

	Equation	$L = L_S + L_n$	$L_{n,max}$	φ_{max}
General	$y'' + B(x) y' + C(x) y = 0$	$\frac{1}{2} f_1(x) y'^2 + \frac{1}{2} f_2(x) y y' + \frac{1}{2} f_3(x) y^2$	$\frac{1}{2} y \left[f_2 y' + \frac{1}{2} f_2' y \right]$	$\frac{1}{4} f_2(x) \cdot y^2(x)$
General Bessel	$B(x) = \frac{\alpha}{x}$ $C(x) = \beta \left(1 + \gamma \frac{\mu^2}{x^2} \right)$	$\frac{1}{2} c_1 \cdot x \left[y'^2 - C(x) y^2 \right] + \frac{1}{2} y \left[f_2 y' + \frac{1}{2} f_2' y \right]$	$\frac{1}{2} c_1 e^{\alpha c_2} \cdot \alpha x^{\alpha - 1} \cdot y \left[\frac{\alpha - 1}{2} x^{-1} y + y' \right]$	$\frac{1}{4} c_1 \cdot e^{\alpha c_2} \cdot \alpha x^{\alpha - 1} \cdot y^2(x)$
Regular Hermite	$B(x) = -x$ $C(x) = x$	$\frac{1}{2} c_1 e^{\frac{-x^2}{2}} \left[y'^2 - x y^2 \right] + \frac{1}{2} y \left[f_2 y' + \frac{1}{2} f_2' y \right]$	$\frac{1}{2} e^{\frac{-x^2}{2}} \cdot y \left[\frac{1}{2} y (x^2 - 1) - x y' \right]$	$\frac{-1}{4} e^{\frac{-x^2}{2}} \cdot x \cdot y^2(x)$
Regular Legendre	$B(x) = \frac{2x}{1 - x^2}$ $C(x) = \frac{l(l+1)}{1 - x^2}$	$\frac{1}{2} c \left\| 1 - x^2 \right\| \left[y'^2 - \left(\frac{l(l+1)}{1 - x^2} \right) \cdot y^2 \right] + \frac{1}{2} y \left[f_2 y' + \frac{1}{2} f_2' y \right]$	$\pm x c y y' \pm \frac{1}{2} c y^2$	$\pm \frac{1}{2} c \cdot x \cdot y^2(x)$

By selecting α, β and γ different (regular, modified, spherical and spherical modified) Bessel equations are obtained.

CONCLUSION

We considered the linear second-order ODEs whose solutions are given by the SF of mathematical physics, and derived general standard and null Lagrangians. The obtained Lagrangians are new and they generalize those previously found. The derived gauge functions are also new. The obtained results are applied to the Bessel, Hermite and Legendre equations, thus, they are of primary interests to physicists and applied mathematicians. The presented results can be easily applied to any ODE with the SF solutions.

ACKNOWLEDGMENTS

We are grateful to two anonymous referees for carefully reading our manuscripts and providing detailed comments and suggestions that helped us to improve our paper.

REFERENCES

1. G.M. Murphy, *Ordinary Differential Equations and Their Solutions*, (Dover Publication, Inc., New York, 2011).

2. H. Hochstadt, *The Functions of Mathematical Physics,* (Dover Publication, Inc., New York, 1986).
3. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, (Elsevier Academic Press, New York, 2005).
4. M.C. Nucci and P.G.L. Leach, J.Math. Phys., **48**, 123510 (2007).
5. Z.E. Musielak, J.Phys. A: Math. Theor., **41**, 055205 (2008).
6. J.L. Cieśliński and T. Nikiciuk, J.Phys. A: Math. Theor., **43**, 175205 (2010).
7. Z.E. Musielak, N. Davachi, and M. Rosario-Franco, Mathematics, **8**, 379 (2020).
8. P.J. Olver, *Applications of Lie Groups to Differential Equations*, (Springer, New York, 1993).
9. D. Krupka and J. Musilova, Diff. Geom. Appl., **9**, 225 (1998).
10. M. Crampin and D.J. Saunders, Diff. Geom. Appl., **22**, 131 (2005).
11. G. Saccomandi and R. Vitolo, J. Math. Sci., **136,** 4470 (2006).
12. Z.E. Musielak and T.B. Watson, Phys. Let. A, **384**, 126838 (2020).
13. P. Caldirola, Nuovo Cim., **18,** 393 (1941).
14. E. Kanai, Prog. Theor. Phys., **3,** 44 (1948).