Noether's Theorem Applied to the Classical and Schrödinger Wave Equations

Nathan Adams[a)], Rebecca Janka[b)], and Johnnie Renee West[c)]

Department of Physics, Southern Nazarene University, Bethany OK 73008

[a)]Corresponding author: nathan3adams@gmail.com
[b)]rjanka@mail.snu.edu
[c)]johnnie.r.west@gmail.com

Abstract: Noether's Theorem, which relates continuous transformations to conservation laws, is applied to the classical wave equation and the Schrödinger equation. Transformations are derived that lead to invariances and conservation laws.

INTRODUCTION

Through the two versions of the Euler-Lagrange equation, Lagrangian dynamics features distinct conservation laws, one for canonical momenta and another for the Hamiltonian. More generally, whenever a functional is both extremal *and* invariant under a continuous transformation, Noether's theorem[1] offers an elegant conservation law expressed as a superposition of these quantities. The applicability of the theorem extends across physics, from classical mechanics to field theories. Since the classical wave equation,

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1}$$

and the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} \tag{2}$$

describe the evolution of their respective wave function $\psi(\mathbf{r},t)$, we seek transformations that leave their functionals invariant in order to find conservation laws. Let us review Noether's theorem.

FUNCTIONALS, EXTREMALS, AND INVARIANCE

Functionals are mappings that take a function as input and produce a real number as output. Let us begin with the type of functional typically encountered in physics, definite integrals of the form

$$\Gamma = \int_a^b L[q^\mu(t), \dot{q}^\mu(t), t]dt \tag{3}$$

where $\mu = 1,2,\ldots,n$, $\dot{q}^\mu \equiv \frac{dq^\mu}{dt}$, and L denotes the Lagrangian. The primary problem in the calculus of variations is to find the set of generalized coordinates $q^\mu(t)$ that make Γ an extremal. Applications include Fermat's Principle and

Hamilton's Principle. As is known[2], the required $q^{\mu}(t)$ satisfy the Euler-Lagrange equation (ELE), which can be written in two forms: either

$$\frac{\partial L}{\partial q^{\mu}} = \dot{p}_{\mu} \tag{4}$$

where p_{μ} denotes the momentum canonically conjugate to q^{μ},

$$p_{\mu} \equiv \frac{\partial L}{\partial \dot{q}^{\mu}}; \tag{5}$$

or alternatively as

$$\frac{\partial L}{\partial t} = -\dot{H}, \tag{6}$$

where H denotes the Hamiltonian, defined by the Legendre transformation

$$H(q^{\mu}, p_{\mu}, t) \equiv \dot{q}^{\nu} p_{\nu} - L(q^{\mu}, \dot{q}^{\mu}, t) \tag{7}$$

(sum repeated indices). These two versions of the ELE offer separate conservation laws: From Eq. (4), p_{μ} is conserved if and only if L contains no explicit dependence on q^{μ}; and by Eq. (6), H is conserved if and only if L does not depend explicitly on t.

As a separate issue, consider a continuously parameterized transformation of the independent variable, $t \to t'$, and of the dependent variables, $q^{\mu} \to q'^{\mu}$:

$$t' = T(t, q^{\nu}, \varepsilon) \tag{8}$$

$$q'^{\mu} = Q^{\mu}(t, q^{\nu}, \varepsilon), \tag{9}$$

where ε is the parameter. A Taylor series expansion of T and Q^{μ} about $\varepsilon = 0$ yields

$$t' = t + \varepsilon\tau(t, q^{\nu}) + O(\varepsilon^2) + \cdots \tag{10}$$

$$q'^{\mu} = q^{\mu} + \varepsilon\zeta^{\mu}(t, q^{\nu}) + O(\varepsilon^2) + \cdots \tag{11}$$

where $\tau = dT/d\varepsilon|_0$ and $\zeta^{\mu} = dQ^{\mu}/d\varepsilon|_0$ are the transformation "generators." The functional Γ is *invariant* if and only if the difference between the functional in the transformed variables, and the functional in the original variables becomes smaller than ε as $\varepsilon \to 0$. More precisely, Γ is said to be invariant if and only if

$$\Gamma' - \Gamma \equiv \int_{a'}^{b'} L\left(q'^{\mu}(t'), \frac{dq'^{\mu}(t')}{dt'}, t'\right) dt' - \int_{a}^{b} L\left(q^{\mu}(t), \frac{dq^{\mu}(t)}{dt}, t\right) dt \ \sim\ \varepsilon^{s} \tag{12}$$

as $\varepsilon \to 0$, where $s > 1$. Under the change of variable $dt' = (dt'/dt)\, dt$ in the first integral, Γ' and Γ can be brought under one integral over dt, and the invariance definition expressed in terms of the integrand as

$$L\left(q'^{\mu}(t'), \frac{dq'^{\mu}(t')}{dt'}, t'\right)\frac{dt'}{dt} - L\left(q^{\mu}(t), \frac{dq^{\mu}(t)}{dt}, t\right) \sim \varepsilon^{s}. \tag{13}$$

Testing for invariance with this definition can be laborious. However, upon differentiating Eq. (13) with respect to ε then setting $\varepsilon = 0$, the Rund-Trautman identity (RTI) results[3,4],

$$\frac{\partial \mathcal{L}}{\partial q^{\mu}}\zeta^{\mu} + p^{\mu}\zeta_{\mu} + \frac{\partial \mathcal{L}}{\partial t}\tau - H\dot{\tau} = 0 \tag{14}$$

or equivalently

$$\left(\frac{\partial L}{\partial q^\mu} - \dot{p}_\mu\right)(\dot{q}^\mu \tau - \zeta^\mu) = \frac{d}{dt}\left[p_\mu \zeta^\mu - H\tau\right], \tag{15}$$

which is necessary and sufficient for invariance. The RTI generalizes to r parameters ε^k for k = 1, 2, ... r, to accommodate combined origin displacements, rotations of axes, and boosts, where to first order in ε, $t' = t + \varepsilon^k \tau_k$ and $q'^\mu = q^\mu + \varepsilon^k \zeta_k^\mu$ (however, we need only $r = 1$ here).

If the functional is invariant *and* extremal, the RTI and ELE hold simultaneously. Substituting the ELE (Eq. 4) into the RTI (Eq. 15) yields the conservation law

$$p_\alpha \zeta_k^\alpha - H\tau_k = const. \tag{16}$$

Noether's theorem produces not only the familiar conservation laws of energy, linear momentum, and angular momentum, but also reveals conservation laws for systems that, at first glance, appear to have none. For instance, the Lagrangian $L(x, \dot{x}, t) = \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right]e^{bt/m}$ produces the equation of motion $m\ddot{x} + b\dot{x} + kx = 0$, a damped oscillator, which conserves neither energy nor momentum. But a combined time translation $t' = t + \varepsilon$ and spatial rescaling $x' = x\left(1 - \varepsilon\frac{b}{2m}\right)$ satisfies the RTI, leading by Eq. (16) to

$$\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{2}b\dot{x}x\right]e^{bt/m} = const. \tag{17}$$

Given a Lagrangian, transformations leading to conservation laws can be found by *imposing* the RTI and solving for the generators. One writes the RTI as a polynomial in coordinate velocity (including using the chain rule for time derivatives of $\tau(q^\mu,t)$ and $\zeta(q^\mu,t)$). Since the RTI must hold *whatever* the velocity, the coefficients of the distinct powers of velocity are set to zero, producing "Killing equations" to be solved for the generators.

Noether's theorem may be extended to fields, which are functions of spacetime coordinates. For our purposes it is sufficient to consider a scalar field $\varphi = \varphi(q^\mu)$, where the q^μ denote n spacetime coordinates. The field's dynamics are described through a functional defined by a multiple integral over a domain $\mathcal{R}$,

$$\Gamma = \int_{\mathcal{R}} \mathcal{L}\, d^n q \tag{18}$$

where the Lagrangian density $\mathcal{L}$ is a function of $\varphi(q^\mu)$, its first derivatives $\frac{\partial \varphi}{\partial q^\mu} \equiv \varphi_\mu$, and the generalized coordinates q^ν. Upon making the functional of Eq. (18) an extremal, the ELE generalizes from Eq. (4) into[4]

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu p^\mu \tag{19}$$

where

$$p^\nu \equiv \frac{\partial \mathcal{L}}{\partial \varphi_\nu} \tag{20}$$

is a field canonical momentum component. The ELE in terms of a Hamiltonian density generalizes from Eq. (6) to

$$\partial_\mu \mathcal{L} = -\partial_\nu \mathcal{H}^\nu_{\ \mu} \tag{21}$$

where

$$\mathcal{H}^\nu_{\ \mu} = \varphi_\mu p^\nu - \delta^\nu_\mu \mathcal{L} \tag{22}$$

is a Hamiltonian tensor component. Let us consider an infinitesimal transformation

$$q'^\mu = q^\mu + \varepsilon\tau^\mu + \cdots \tag{23}$$

$$\varphi' = \varphi + \varepsilon\zeta + \cdots. \tag{24}$$

The field-theory version of the RTI becomes a generalization of Eq. (14),[4]

$$\frac{\partial\mathcal{L}}{\partial\varphi}\zeta + \frac{\partial\mathcal{L}}{\partial\varphi_\mu}\zeta_\mu + \frac{\partial\mathcal{L}}{\partial q^\mu}\tau^\mu - \mathcal{H}^\mu{}_\nu\tau^\nu{}_\mu = 0 \tag{25}$$

where $\zeta_\mu = \partial\zeta/\partial q^\mu$ and $\tau^\nu{}_\mu = \partial\tau^\nu/\partial q^\mu$. Thanks to the product rule for derivatives, Eq. (25) may be alternatively written as

$$\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\nu p^\nu\right](\varphi_\mu\,\tau^\mu - \zeta) = \partial_\mu[\,p^\mu\zeta - \mathcal{H}^\mu{}_\nu\tau^\nu], \tag{26}$$

a generalization of Eq. (15). When the functional is invariant *and* extremal, the ELE and RTI hold simultaneously and Eq. (26) becomes an equation of continuity,

$$\partial_\mu j^\mu = 0 \tag{27}$$

where

$$j^\mu = p^\mu\zeta - \mathcal{H}^\mu{}_\nu\tau^\nu. \tag{28}$$

Integrating Eq. (27) over the x-axis yields, in the case of two-dimensional spacetime (t,x),

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(p^0\zeta - \mathcal{H}^0{}_\nu\tau^\nu)dx + [p^1\zeta - \mathcal{H}^1{}_\nu\tau^\nu]\,|_{-\infty}^{+\infty} = 0. \tag{29}$$

Assuming the field vanishes at spatial infinity, Eq. (29) becomes

$$\int_{-\infty}^{+\infty}(p^0\zeta - \mathcal{H}^0{}_\nu\,\tau^\nu)\,dx = const. \tag{30}$$

Examples in four-dimensional spacetime include global gauge transformations in electrodynamics, leading to local charge conservation. In field theory, as in particle mechanics, given a Lagrangian density one may impose the RTI to find generators that make the functional extremal, producing conservation laws expressed as equations of continuity.

Towards this end, one writes the canonical momenta and the Hamiltonian in terms of "velocities," derivatives of fields with respect to spacetime coordinates. As a polynomial in "velocity" components, the RTI must hold *whatever* the "velocities." The coefficients of distinct powers of "velocity" are set to zero, producing Killing equations to be solved for the generators. We apply this program to the classical inhomogeneous linear wave equation and the time-dependent Schrödinger equation.

Noether's Theorem and the Linear Wave Equation

Consider the vertical component of **F** = m**a** applied to an increment of guitar string having mass dm, linear density μ, and tightened to tension T. For a segment of string of length ds displaced a distance y (the field φ) from its undisturbed position (the x axis), and with a damping force proportional to vertical velocity, Newton's second law gives

$$T\,\Delta(\sin\theta) - \eta\,dm\frac{\partial y}{\partial t} - g dm = dm\frac{\partial^2 y}{\partial t^2} \tag{31}$$

where g denotes the gravitational field, η a damping coefficient, and θ the segment's inclination above the horizontal at its endpoints. Using $dm = \mu ds$, and noting that for small angles $ds \approx dx$ and $\sin\theta \approx \tan\theta = \partial y/\partial x$, in the limit as $dx \to 0$, Eq. (31) becomes a linear inhomogeneous wave equation,

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\lambda}{v}\frac{\partial y}{\partial t} + \gamma \tag{32}$$

where $v = \sqrt{T/\mu}$ is the wave's translational velocity down the x axis, $\lambda = \eta/v$, and $\gamma = g/v^2$. Letting $s = vt$, the Lagrangian density

$$\mathcal{L} = (\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}y'^2 - y\gamma)e^{\lambda s} \tag{33}$$

(where $\dot{y} \equiv \frac{\partial y}{\partial s} \equiv y_0$ and $y' \equiv \frac{\partial y}{\partial x} \equiv y_1$), when substituted into Eq. (19), gives Eq. (32).

It is instructive to examine the physical interpretations of the Lagrangian density terms. Consider an increment of string of length $ds \approx dx$, and recall $\mu = T/v^2$ and $\gamma = g/v^2$. The increment's kinetic energy is

$$dK = \frac{1}{2}dm\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\frac{T dx}{v^2}\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}T\left(\frac{\partial y}{\partial s}\right)^2 dx; \tag{34}$$

its gravitational potential energy is

$$dU_{grav} = (dm)gy = \mu(dx)gy = T\gamma y\, dx; \tag{35}$$

and for the elastic potential energy, with "spring constant" $T \sin\theta/\Delta y$, we have

$$dU_{el} = \frac{1}{2}\frac{T\left(\frac{\partial y}{\partial x}\right)}{\Delta y}(\Delta y)^2 = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{36}$$

Now the Lagrangian density is recognized physically as

$$\mathcal{L} = \frac{e^{\lambda s}}{T}\frac{d}{dx}\left(K - U_{el} - U_{grav}\right). \tag{37}$$

The canonical momenta are

$$p_0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}} = y_0 e^{\lambda t} \tag{38}$$

and

$$p_1 \equiv \frac{\partial \mathcal{L}}{\partial y'} = -y_1 e^{\lambda t}. \tag{39}$$

Eq. (22) gives the Hamiltonian tensor components. Formally, they are functions of canonical momenta, but to extract the Killing equations from the RTI we must write the tensor components in terms of the "velocities," and thereby obtain

$$\mathcal{H}^{\nu}{}_{\mu} = \begin{pmatrix} \mathcal{H}^0_0 & \mathcal{H}^0_1 \\ \mathcal{H}^1_0 & \mathcal{H}^1_1 \end{pmatrix} = e^{\lambda s}\begin{pmatrix} \xi + \gamma y & -y_0 y_1 \\ -y_0 y_1 & -\xi + \gamma y \end{pmatrix} \tag{40}$$

where $\xi = \frac{1}{2}y_0^2 + \frac{1}{2}y_1^2$. Under an infinitesimal transformation

$$t' = t + \varepsilon\tau^0, \tag{41}$$

$$x' = x + \varepsilon\tau^1, \tag{42}$$

$$y = y + \varepsilon\zeta, \tag{43}$$

and imposing the RTI, Eq. (25) becomes

$$\begin{aligned} -\gamma[\zeta + y(\lambda\tau^0 + \tau^0_0 + \tau^1_1)] + y_0\zeta_0 - y_1\zeta_1 + \frac{{y_0}^2}{2}(\lambda\tau^0 - \tau^0{}_0 + \tau^1{}_1) \\ + \frac{{y_1}^2}{2}(-\lambda\tau^0 - \tau^0{}_0 + \tau^1{}_1) + y_0 y_1(\tau^1{}_0 + \tau^0{}_1) \quad = 0. \end{aligned} \tag{44}$$

Since this expression must hold whatever y_0 and y_1 may be, we set to zero the coefficients of distinct derivatives of y, producing the Killing equations:

no-derivative coefficient: $\gamma[\zeta + y(\lambda\tau^0 + \tau^0{}_0 + \tau^1{}_1)] = 0$ (K1)

y_o coefficient: $\zeta_0 = 0$ (K2)

y_1: $\zeta_1 = 0$ (K3)

$y_o{}^2$: $\lambda\tau^0 = \tau^0{}_0 - \tau^1{}_1$ (K4)

$y_1{}^2$: $-\lambda\tau^0 = \tau^0{}_0 - \tau^1{}_1$ (K5)

$y_o y^1$: $\tau_1^0 = -\tau_0^1.$ (K6)

Eqs. (K2-K3) show that, at most, $\zeta = \zeta(y)$. Eq. (K6) suggests a separation of variables, $\tau_1^0 = -\tau_0^1 = \kappa$ where $\kappa = const.$ This integrates to

$$\tau^0 = \kappa x + f(s) \tag{45}$$

$$\tau^1 = -\kappa s + g(x) \tag{46}$$

where f and g are arbitrary functions. Comparing Eqs. (K4) and (K5) gives $\lambda\tau^0 = 0$, and thus $\tau_0^0 = \tau_1^1$, which leads to another separation of variables, $\frac{df}{ds} = \frac{dg}{dx} = C$, where $C = const.$, and therefore

$$\tau^0 = Cs + \kappa x + a \tag{47}$$

$$\tau^1 = -\kappa s + Cx + b \tag{48}$$

where a and b are integration constants.

Now Eq. (K1) becomes

$$\gamma(\zeta + 2Cy) = 0 \tag{49}$$

which presents alternatives depending on whether γ vanishes.

(a) If $\gamma \neq 0$, then $\zeta = -2Cy$, and using Eqs. (K2-K3),

$$\dot{\zeta} = \zeta_0 + \zeta_1\dot{y} = 0 = -2C\dot{y}, \tag{50}$$

which means $C = 0$ and thus $\zeta = 0$, which also takes C out of Eqs. (47-48).

(b) If $\gamma = 0$ then Eq. (49) puts no further constraint on ζ, although Eqs. (K2-K3) still hold; together these allow $\zeta = const. \equiv \zeta_o$ with nonzero C.

Among the possibilities of cases (a) and (b), we consider the transformations

$$t' = t + \varepsilon(\delta_{\gamma 0}Cs + \kappa x + a) \tag{51}$$

$$x' = x + \varepsilon(\delta_{\gamma 0}Cx - \kappa s + b) \tag{52}$$

$$y' = y + \varepsilon\zeta_o\delta_{\gamma 0} \tag{53}$$

where $\delta_{\gamma 0} = 0$ if $\gamma \neq 0$, and $\delta_{\gamma 0} = 1$ if $\gamma = 0$. When the functional is both extremal and invariant under Eqs. (51)-(53), Eq. (27) gives

$$\partial_\mu(p^\mu\zeta - \mathcal{H}^\mu{}_\nu\tau^\nu) = 0 \tag{54}$$

and Eq. (30) implies

$$\int_{-\infty}^{+\infty}(p^0\zeta - \mathcal{H}^0{}_\nu\tau^\nu)\,dx = const., \tag{55}$$

which in our case becomes

$$\int_{-\infty}^{+\infty}[y_0 e^{\lambda t}\zeta_o\delta_{\gamma 0} - (\xi + \gamma y)(Cs + \kappa x + a) + y_0 y_1(Cx - \kappa s + b)]dx = const. \tag{56}$$

Consider a time translation, where $a \neq 0$ but C, κ, b and ζ_o vanish. Eq. (56) reduces to

$$\int_{-\infty}^{+\infty}(\xi + \gamma y)\, dx = const. \tag{57}$$

Recognizing the integrand as mechanical energy density, invariance under a time translation implies energy conservation.

Under a vertical displacement of the wave, $a = b = 0$ and $C = \kappa = 0$ but $\zeta_o \neq 0$ (which implies $\gamma = 0$). Eq. (56) gives

$$e^{\lambda t}\int_{-\infty}^{+\infty} y_0 dx = const. \tag{57}$$

Multiplying by the mass linear density μ, Eq. (57) becomes

$$\int_{-\infty}^{+\infty} y_0 dm = const.\times e^{-\lambda} \ . \tag{58}$$

The integral describes the total vertical momentum as the string vibrates in the y-direction. If $\lambda \neq 0$ the vertical momentum damps with time, analogous to Eq. (17). But if $\lambda = 0$ then Eq. (57) becomes

$$\int_{-\infty}^{+\infty} y_0 dm = const. \tag{59}$$

which says that, with negligible gravity and no damping, the vertical momentum carried by the wave on the entire string is conserved. Of course, due to the elastic restoring force, the instantaneous momentum of any *increment* of string is not conserved, but for any bit of string moving upwards in one place, another bit of string elsewhere at the same time moves downward with the same speed, allowing the y-momentum of the *entire string* to remain constant.

Noether's theorem and the Schrödinger equation

The Schrödinger equation for a particle of mass m interacting with a potential energy $U(x,t)$ is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} \tag{60}$$

where $\psi = \psi(t,x)$ and $\hbar$ is the reduced Planck's constant. The Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi_\mu^*}\right) \tag{62}$$

(sum repeated indices with μ = 0, 1) coincides with Eq. (2) if the Lagrangian density $\mathcal{L}$ has the form

$$\mathcal{L}(\psi,\psi^*,\psi_0,\psi_0^*,\psi_1,\psi_1^*,t,x) = -\frac{\hbar^2}{2m}\psi_1^*\psi_1 - \frac{\hbar}{2i}(\psi^*\psi_0 - \psi_0^*\psi) - \psi^* U\psi \tag{63}$$

where $x_0 = t$, $x_1 = x$, $\partial\psi/\partial x_\mu = \psi_\mu$ and * indicates the complex conjugate (*cc*). For brevity let $\alpha = -\hbar^2/2m$ and $\beta = \hbar i/2$. From Eq. (6) the canonical momenta p_μ are:

$$p_0 = \frac{\partial\mathcal{L}}{\partial\psi_0} = \beta\psi^*, \qquad p_0^* = \frac{\partial\mathcal{L}}{\partial\psi_0^*} = -\beta\psi \tag{64}$$

$$p_1 = \frac{\partial\mathcal{L}}{\partial\psi_1} = \alpha\psi^*, \qquad p_1^* = \frac{\partial\mathcal{L}}{\partial\psi_1^*} = \alpha\psi. \tag{65}$$

From Eq. (22) the Hamiltonian tensor components are:

$$\{\mathcal{H}^{\mu}{}_{\nu}\} = \begin{pmatrix} \mathcal{H}^{0}{}_{0} & \mathcal{H}^{0}{}_{1} \\ \mathcal{H}^{1}{}_{0} & \mathcal{H}^{1}{}_{1} \end{pmatrix} = \begin{pmatrix} [-\alpha\psi_1^*\psi_1 + \psi^* U\psi] & \beta(\psi^*\psi_1 - \psi\psi_1^*) \\ \alpha(\psi_1^*\psi_0 + \psi_1\psi_0^*) & [\alpha\psi_1^*\psi_1 - \beta(\psi^*\psi_0 - \psi_0^*\psi) + \psi^* U\psi] \end{pmatrix}. \tag{66}$$

Consider an infinitesimal transformation parameterized by ε:

$$t' = t + \varepsilon\tau^0 \tag{67}$$

$$x' = x + \varepsilon\tau^1 \tag{68}$$

$$\psi' = \psi + \varepsilon\zeta \tag{69}$$

$$\psi^{*\prime} = \psi^* + \varepsilon\zeta^* \tag{70}$$

with generators $\tau^\mu = \tau^\mu(t,x)$, $\zeta = \zeta(t,x)$, $\zeta^* = \zeta^*(t,x)$. The system is invariant if and only if the invariance identity holds, which in this instance requires:

$$\frac{\partial\mathcal{L}}{\partial\psi}\zeta + \frac{\partial\mathcal{L}}{\partial\psi^*}\zeta^* + p^\mu(\partial_\mu\zeta) + p^{*\mu}(\partial_\mu\zeta^*) + (\partial_\mu\mathcal{L})\tau^\mu - \mathcal{H}^{\mu}{}_{\nu}(\partial_\mu\tau^\nu) = 0. \tag{71}$$

If the functional is invariant *and* extremal, Noether's theorem yields an equation of continuity, Eq. (27).

We impose the invariance identity and solve for generators. Since the invariance identity must hold *whatever* $\psi(t,x)$ and $\psi^*(t,x)$ happen to be, the coefficients of their distinct derivatives are set to zero, to produce the Killing equations. With primes denoting $\partial/\partial x$ and overdots denoting $\partial/\partial t$, they are:

no derivative coefficient: $(\psi^*\zeta + \psi\zeta^*)U - (\psi^*\dot{\zeta}\beta + \psi\dot{\zeta}^*\beta^*) + \psi^*\psi\,\partial_\mu(U\tau^\mu)$ (Ka)

ψ_1 *and cc*: $\alpha\zeta' + \beta\dot{\tau}_1\psi = 0$ (Kb)

ψ_0 *and cc*: $\zeta + \tau'^1\psi = 0$ (Kc)

$\psi_1^*\psi_0$: $\tau'^0 = 0$ (Kd)

$\psi_1^*\psi_1$: $\dot{\tau}^0 = \tau'^1$ (Ke).

Eq. (Kd) tells us $\tau^0 = \tau^0(t)$, which turns (Ke) into

$$\frac{d\tau^0}{dt} = \frac{\partial\tau^1}{\partial x}. \tag{72}$$

Since, in general, $\tau^1 = \tau^1(t,x)$, a separation constant *A* is not the most general solution to Eq. (70). But this special case in Eq. (72) gives

$$\tau^0 = At + a \tag{73}$$

$$\tau^1 = Ax + g(t). \tag{74}$$

Placing Eq. (74) into (Kc) gives

$$\zeta = -A\psi, \tag{75}$$

and putting Eq. (74) into (Kb) yields

$$\frac{\hbar}{m}\zeta' = i\dot{g}\psi. \tag{76}$$

Consistency between Eqs. (75) and (76) puts a constraint on ψ of the form $\psi' = -\frac{i\dot{g}m}{\hbar A}\psi$, so that

$$\psi \sim e^{-i\dot{g}mx/\hbar A}. \tag{77}$$

This suggests a Fourier component of a wave function corresponding to a particle moving along the *x*-axis, for which

$$\psi(t,x) = \int_{-\infty}^{+\infty} \phi(k) e^{i(kx \pm \omega t)} \frac{dk}{2\pi}. \tag{78}$$

Therefore, by comparing Eqs. (77) and (78), we set $k = -\frac{\dot{g}m}{\hbar A}$, which gives

$$g(t) = -\frac{\hbar k}{m} At + b \tag{79}$$

where *b* is an integration constant. We notice that $\hbar k$ is the particle's momentum. In summary, this solution of the Killing equations is

$$\tau^0 = At + a \tag{80}$$

$$\tau^1 = A\left(x - \frac{\hbar k}{m} t\right) + b \tag{81}$$

$$\zeta = -A\psi. \tag{82}$$

The remaining Killing equation (Ka) constrains the potential energy for conservation laws to follow. With Eqs. (80)-(82), it requires

$$2A(\psi^*\psi)U - A\frac{\hbar i}{2}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) + \psi^*\psi\,[\partial_0(U\tau^0) + \partial_1(U\tau^1)] = 0. \tag{83}$$

But even for a free particle for which $U = 0$, the middle term will not vanish unless $A = 0$. Henceforth we set $A = 0$, and thus the conserved quantity of Eq. (30) becomes

$$\int_{-\infty}^{+\infty} [\mathcal{H}^0{}_0\tau^0 + \mathcal{H}^0{}_1\tau^1] dx = const. \tag{84}$$

which is

$$\int_{-\infty}^{+\infty} \left[\left(-\frac{\hbar^2}{2m}\psi_1^*\psi_1 + \psi^* U\psi\right)\tau^0 - \frac{\hbar i}{2}(\psi^*\psi_1 - \psi\psi_1^*)\tau^1\right] dx = const. \tag{85}$$

An integration by parts on the first term, $\partial_1(\psi^*\psi_1) = \psi_1^*\psi_1 + \psi^*\psi_{11}$, plus the requirement that the wave function vanishes at infinity, turns Eq. (85) into

$$\int_{-\infty}^{+\infty} \left[\psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U\psi\right)\tau^0 - \frac{\hbar i}{2}(\psi^*\psi_1 - \psi\psi_1^*)\tau^1\right] dx = const. \tag{86}$$

With $A = 0$ the transformations of Eqs. (80)-(82) describe only time and space translations:

$$t' = t + \varepsilon a \tag{87}$$

$$x' = x + \varepsilon b\,. \tag{88}$$

$$\psi' = \psi \tag{89}$$

and the constraint of Eq. (83), when integrated over all space, reduces to

$$a\left\langle\frac{\partial U}{\partial t}\right\rangle + b\left\langle\frac{\partial U}{\partial x}\right\rangle = const. \tag{90}$$

where the brackets denote expectation values. Under a time translation, for which $a \neq 0$ but $b = 0$, if $\langle \partial U/\partial t\rangle = 0$, from Eq. (86) the conservation of the energy results:

$$\int_{-\infty}^{+\infty} \psi^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U\psi\right) dx = \left\langle \frac{p^2}{2m} + U \right\rangle = \ const. \tag{91}$$

Under a spatial translation with $b \neq 0$ but $a = 0$, if $\langle \partial U/\partial x \rangle = 0$ the conservation of probability results[5]:

$$\frac{\hbar i}{2m} \int_{-\infty}^{+\infty} (\psi^* \psi_1 - \psi\psi_1^*)\, dx = const. \tag{92}$$

Comment

These results for the linear wave equation and the Schrödinger equation tell us little about their conservation laws that was not already known from other perspectives. However, our exercise illustrates the powerful generality of Noether's theorem, and provides consistency checks on those other perspectives.

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