

Generalized Non-Standard Lagrangians

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Abstract. A generalized Lagrange formalism is developed for Ordinary Differential Equations (ODE) with the special function solutions [1]. The formalism is based on non-standard Lagrangians, which represent a novel family of Lagrangians. It is shown that the Euler-Lagrange equation needs to be supplemented with an auxiliary condition to retrieve the original equation - this is a new phenomenon in the calculus of variations.

INTRODUCTION

Consider differential equations of the form $y''(x) + B(x)y'(x) + C(x)y(x) = 0$, where $B(x)$ and $C(x)$ are ordinary smooth functions of x . Note that many important equations of mathematical physics, such as Bessel, modified Bessel, spherical Bessel, Legendre, associated Legendre, Laguerre, associated Laguerre, Hermit, Chebyshev, Jacobi, etc. [2], can be derived from the above ODE by proper choices of $B(x)$ and $C(x)$. In general, there are at least three different methods to obtain the ODE, namely, by the method of separation of variables in the wave, Helmholtz, Laplace and other Partial Differential Equations (PDEs), by using the Lagrangian formalism, and by the so-called Lie group method. The separation of variables method is the most commonly used in mathematical physics and applied mathematics. The Lagrange formalism is already well-established for the so-called standard Lagrangians [3]. Finally, the Lie group approach uses irreducible representations of Lie groups that correspond to any special function ODE [4]. The main disadvantage of the last two methods is the requirement of knowledge of Lagrangian for each ODE or Lie group associated with such ODE. Neither is easy to be determined. In this paper, we concentrate on the Lagrange formalism and present a general method of deriving the required non-standard Lagrangians.

The Lagrangian formalism is commonly used in modern classical and quantum physics, and the principle that underlies this formalism is the principle of least action or Hamilton's Principle. The main aim of this paper is to extend the formalism to ODEs whose solutions are the special functions of mathematical physics by using the so-called non-standard Lagrangians. The obtained results are important as they show that the calculus of variations must be modified by setting a new condition called the auxiliary condition, and that without this condition the formalism does not allow deriving the original ODE. We apply our formalism to the Bessel equation and present its new non-standard Lagrangian. Our choice of the equation is justified by the fact that the Bessel equation is used in some physical applications, for example, describing different wave motions, and therefore the result of such application should be of interest to undergraduate and graduate science students, as well as to physicists, applied mathematicians and engineers who use this equation in their work.

Calculus of Variations, Euler-Lagrange Equation, and Lagrangians

In order to find maxima and minima of functionals that are given as integrals, the calculus of variations uses small changes to these functionals known as variations. Functions that maximize or minimize these functionals can be found by solving the Euler-Lagrange equation. Fermat's theorem in calculus, which states that when a local extremum is achieved by a function its derivative is zero at that point, is analogous to the Euler-Lagrange equation in the calculus of variations.

The dynamics of a system is determined by a functional called the Lagrangian of the system. Lagrangians are one of the most sophisticated ways of formulating theoretical physics. Lagrangians are advantageous when including additional forces, studying the stability of solutions, applying perturbation theory, establishing the existence of resonances, and calculating Lyapunov exponents. Historically, Lagrangians do satisfy the Euler-Lagrange equation, however, our results show that for our new class of non-standard Lagrangians, Euler-Lagrange needs to be supplemented with an auxiliary condition.

For a Lagrangian to be called standard, it is required that the terms of the Lagrangian are identified as the kinetic and the potential energy terms. However, if there is a Lagrangian whose terms cannot be related to the well-known kinetic and potential energy terms, then the Lagrangian is called non-standard (NSL) and it still has the same mathematical properties as its standard counterpart. Among various applications, Alekseev and Arbuzov (1984) used the NSLs to formulate the Yang-Mills field theory [5], which is responsible for our basic understanding of the Standard Model of particle physics [6], and hence demonstrated the usefulness of NSLs for fundamental theories of modern physics and theoretical physics.

Standard and non-standard Lagrangians form two distinct families of Lagrangians, and possible mathematical and physical relationships between them will be explored and discussed in another paper. The non-standard Lagrangians should be treated as new generating functionals for ODEs for which standard Lagrangians are already known [7]. A method to derive such non-standard Lagrangians for the ODEs of mathematical physics is developed in this paper.

Non-Standard Lagrangians

Proposition 1

Functions $f(x)$, $g(x)$, $h(x)$ described below exist for the following NSL such that this new NSL gives us the general ODE of $y''(x) + B(x)y'(x) + C(x)y(x) = 0$ [8]. Our new general L_{NSL} is the following

$$L_{NSL} = \frac{1}{f(x)y'(x)+g(x)y(x)+h(x)} \quad (1)$$

where

$$f(x) = v(x)^3 e^{\int^x 2B(t)dt} \quad (2)$$

$$g(x) = -v'(x)v(x) e^{\int^x 2B(t)dt} \quad (3)$$

$$h(x) = v^2(x) e^{\int^x B(t)dt} \quad (4)$$

such that $v(x)$ is a solution to the original equation.

Proof

Substituting L_{NSL} into the Euler-Lagrange (E-L) equation

$$\frac{d}{dx} \left(\frac{dL_{NSL}}{dy'} \right) - \frac{dL_{NSL}}{dy} = 0 \quad (5)$$

we obtain the following set of three ODEs for the functions $f(x)$, $g(x)$, and $h(x)$. Solutions to the following three differential equations yield $f(x)$, $g(x)$, and $h(x)$, and the generalized L_{NSL} is obtained as desired

$$\frac{3}{2} \frac{g(x)}{f(x)} + \frac{1}{2} \frac{f'(x)}{f(x)} = B(x) \quad (6)$$

$$\frac{\frac{1}{2}g^2(x)}{f^2(x)} + \frac{g'(x)}{f(x)} - \frac{\left(\frac{1}{2}f'(x)g(x)\right)}{f^2(x)} = C(x) \quad (7)$$

$$\frac{\frac{1}{2}g(x)h(x)}{f^2(x)} + \frac{h'(x)}{f(x)} - \frac{\frac{1}{2}f'(x)h(x)}{f^2(x)} = 0. \quad (8)$$

In order so solve this set of equations, we define $u(x) = \frac{f'(x)}{f(x)}$ which yields the following Riccati equation

$$u'(x) + \frac{1}{3} u^2(x) - \frac{1}{3} B(x)u(x) - \left[\frac{2}{3} B^2(x) + 2B'(x) - 3C(x)\right] = 0. \quad (9)$$

To solve this Riccati equation, we use the $u(x) = 3 \frac{w'(x)}{w(x)}$ transformation to obtain the following second order differential equation

$$w''(x) - \frac{1}{3} B(x)w'(x) - \frac{1}{3} \left[\frac{2}{3} B^2(x) + 2B'(x) - 3C(x)\right]w(x) = 0. \quad (10)$$

By solving Eq. (10), we find $f(x)$, $g(x)$, $h(x)$ to construct our generalized L_{NSL} , which is obtained when these functions are substituted into Eq. (1). This concludes the proof.

Proposition 2

The following auxiliary condition to the E-L equation is needed in order to retrieve the original equation back

$$v''(x) + B(x)v'(x) = -C(x)v(x). \quad (11)$$

Proof

We substitute the obtained L_{NSL} into the E-L equation (Eq. (5)) to get our original ODE back but instead we obtain the following equation

$$-v(x)y''(x) - B(x)v(x)y'(x) + [B(x)v(x) + V''(x)]y(x) = 0 \quad (12)$$

and this equation does not allow for the reproduction of the original equation on its own. However, supplementing this equation by Eq. (11), we obtain the original ODE as desired

$$y''(x) + B(x)y'(x) + C(x)y(x) = 0. \quad (13)$$

This concludes the proof.

Non-Standard Lagrangians for the Bessel Equation

For the Bessel equation, we have $B(x) = \frac{1}{x}$ and $C(x) = 1 - \frac{\mu^2}{x^2}$ where μ is either real or integer constant. The above procedure allows finding the functions $f(x)$, $g(x)$, and $h(x)$. Therefore, the non-standard Lagrangian for the Bessel is

$$L_{NSL}^B = \frac{1}{v^3 y' e^{2\ln|x|} - v v' y e^{2\ln|x|} + v^2 e^{2\ln|x|}} \quad (14)$$

where $v(x)$ is given by one of the solutions of Bessel Equation. These solutions are power series expansions and they are Bessel functions; the solutions are typically denoted as $J_\mu(x)$ and $J_{-\mu}(x)$ for μ being real and as $J_\mu(x)$ and $Y_\mu(x)$ for μ being an integer. This dependence of non-standard Lagrangians directly on one of the Bessel solutions is a new phenomenon in the calculus of variations.

CONCLUSION

We considered second order linear ordinary differential equations with non-constant coefficients. We developed a new class of non-standard Lagrangians for these equations. We also showed that an auxiliary condition to the calculus of variations is required to obtain the original equation; this is a new phenomenon in the calculus of variations.

In our application, we presented the first non-standard Lagrangian for the Bessel equation. Since the standard Lagrangian for the equation is already known [7], the obtained non-standard Lagrangian can be of interest to physicists, applied mathematicians and engineers who use the Bessel equation.

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